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802 Homework 3

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## Problem 1

Show that the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}^T\mathbf{X} & \mathbf{T}^T \\ \mathbf{T} & \mathbf{0} \end{bmatrix}$$

is a non-singular matrix, where $\mathbf{X}$ is the design matrix for a non-full rank linear model and $\mathbf{T}$ is a corresponding matrix such that $\mathbf{T}\boldsymbol{\beta} = \mathbf{0}$ is a side condition.

**Solution:** Assume that $\mathbf{X} \in \mathbb{R}^{n\times p}$ with $\text{rank}(X) = k < p \leq n$. Then, since $\mathbf{T}\boldsymbol{\beta} = \mathbf{0}$ is a side condition, by definition we have $\text{rank}(\mathbf{T}) = p - k$ the following two properties

$$\begin{aligned} 1) \quad & \text{rank}\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \end{pmatrix} = p \\ 2) \quad & \text{rank}\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \end{pmatrix} = \text{rank}(\mathbf{X}) + \text{rank}(\mathbf{T}) \end{aligned}$$

which implies that the rows of $\mathbf{T}$ and $\mathbf{X}$ are linearly independent. Therefore, the columns of $\mathbf{T}^T$ and $\mathbf{X}^T$ are linearly independent. Since $C(\mathbf{X}^T) = C(\mathbf{X}^T\mathbf{X})$, the columns of $\mathbf{T}^T$ and $\mathbf{X}^T\mathbf{X}$ are linearly independent. Thus, $\mathbf{A}$ is invertible.

## Problem 2

Define $\mathbf{H} := \mathbf{X}^T\mathbf{X} + \mathbf{T}^T\mathbf{T}$. Show that the inverse of $\mathbf{A}$ in Problem 1 is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{H}^{-1}\mathbf{X}^T\mathbf{X}\mathbf{H}^{-1} & \mathbf{H}^{-1}\mathbf{T}^T \\ \mathbf{T}\mathbf{H}^{-1} & \mathbf{0} \end{bmatrix}.$$

**Solution:** Was out sick late last week, couldn't seek help on this one =(.

## Problem 3

In the non-full rank linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, let $\mathbf{C} \in \mathbb{R}^{m\times p}$ be a full-row-rank matrix such that $\mathbf{C}\boldsymbol{\beta}$ is estimable. Define $SSH = \big(\mathbf{C}\widehat{\boldsymbol{\beta}}\big)^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\big(\mathbf{C}\widehat{\boldsymbol{\beta}}\big)$ and $SSE = \mathbf{y}^T\big(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}^T\big)\mathbf{y}$, where $\mathbf{G}$ is a generalized inverse of $\mathbf{X}^T\mathbf{X}$. Show that $SSH$ and $SSE$ are (statistically) independent.

**Solution:** First notice that we can express $\widehat{\boldsymbol{\beta}}$ as

$$\widehat{\boldsymbol{\beta}} = \mathbf{G}\mathbf{X}^T\mathbf{y}.$$

Therefore, we can rewrite $SSH$ in the following way:

$$SSH = \big(\mathbf{C}\widehat{\boldsymbol{\beta}}\big)^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\big(\mathbf{C}\widehat{\boldsymbol{\beta}}\big) = \mathbf{y}^T\mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\mathbf{G}\mathbf{X}^T\mathbf{y} \equiv \mathbf{y}^T\mathbf{A}\mathbf{y}$$

where $\mathbf{A}$ is the matrix given by

$$\mathbf{A} = \mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\mathbf{G}\mathbf{X}^T.$$

Also, denote $\mathbf{B} = \mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}^T$ and so $SSE = \mathbf{y}^T\mathbf{B}\mathbf{y}$. To show that $SSH$ and $SSE$ are independent, it suffices to show that $\mathbf{A}\mathbf{B} = \mathbf{0}$ by a result of quadratic forms of normal random variables. Indeed,

$$\begin{aligned}
\mathbf{A}\mathbf{B} &= \mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\mathbf{G}\mathbf{X}^T\Big(\mathbf{I} - \mathbf{X}\mathbf{G}\mathbf{X}^T\Big) = \mathbf{A} - \mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\mathbf{G}\mathbf{X}^T\mathbf{X}\mathbf{G}\mathbf{X}^T \\
&= \mathbf{A} - \mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\big(\mathbf{X}^T\mathbf{X}\big)^{-}\mathbf{X}^T\mathbf{X}\big(\mathbf{X}^T\mathbf{X}\big)^{-}\mathbf{X}^T \\
&= \mathbf{A} - \mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\big(\mathbf{X}^T\mathbf{X}\big)^{-}\mathbf{X}^T \\
&= \mathbf{A} - \mathbf{X}\mathbf{G}^T\mathbf{C}^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\mathbf{C}\mathbf{G}\mathbf{X}^T = \mathbf{A} - \mathbf{A} = \mathbf{0}.
\end{aligned}$$

Therefore, $SSH$ and $SSE$ are independent.

## Problem 4

For the non-full rank version of the Gauss-Markov model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

derive the $F$-test for testing $H_0\colon \mathbf{C}\boldsymbol{\beta} = \mathbf{t}$, where $\mathbf{C}$ is a $q \times p$ matrix of rank $q$, $\mathbf{t}$ is a specified vector of constants, and the vector of parametric functions $\mathbf{C}\boldsymbol{\beta}$ is estimable.

**Solution:** Consider the equivalent test $H_0\colon \mathbf{C}\boldsymbol{\beta} - \mathbf{t} = \mathbf{0}$ and now

$$SSH = \big(\mathbf{C}\widehat{\boldsymbol{\beta}} - \mathbf{t}\big)^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\big(\mathbf{C}\widehat{\boldsymbol{\beta}} - \mathbf{t}\big),$$

where $\mathbf{G}$ is a generalized inverse of $\mathbf{X}^T\mathbf{X}$ as before. Notice that

$$\mathbf{C}\widehat{\boldsymbol{\beta}} - \mathbf{t} = \mathbf{C}\mathbf{G}\mathbf{X}^T\mathbf{y} - \mathbf{t} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\mu = \mathbf{C}\boldsymbol{\beta} - \mathbf{t}$ and $\boldsymbol{\Sigma} = \sigma^2\mathbf{C}\mathbf{G}\mathbf{C}^T$. Therefore, it follows that

$$\frac{SSH}{q} \sim \chi^2_q(\lambda)$$

where the noncentrality parameter $\lambda$ is given by

$$\lambda = \big(\mathbf{C}\boldsymbol{\beta} - \mathbf{t}\big)^T\big(\mathbf{C}\mathbf{G}\mathbf{C}^T\big)^{-1}\big(\mathbf{C}\boldsymbol{\beta} - \mathbf{t}\big)$$

and is zero under the null hypothesis. Recall $SSE$ and $\widehat{\boldsymbol{\beta}}$ are independent. Therefore, since this $SSH$ is just a function of $\widehat{\boldsymbol{\beta}}$, it is independent of $SSE$. Also recalling that

$$\frac{SSE}{n-k-1} \sim \chi^2_{n-k-1}$$

it now follows that we can test the hypothesis using the $F$ statistic

$$F = \frac{SSH/q}{SSE/(n-k-1)} \sim F_{q,n-k-1}.$$

## Problem 5

Consider the effects version of the one-way layout model given by

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \quad j = 1, ..., n_i, \;\; i = 1, ..., a,$$

with $\sum_{i=1}^a n_i > a$. Explain which of the following linear parametric functions are estimable.

(a) $\mu + \alpha_1$

(b) $\mu$

(c) $\sum_{i=2}^a \alpha_i$

(d) $\alpha_2 - \alpha_a$

**Solution:** First, we should notice that the design matrix here can be written as

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_a \end{pmatrix}$$

where $\mathbf{x}_i$ is a row vector of all zeros, except for a 1 in the first index (for $\mu$) and also index $i+1$ (for $\alpha_i$).

(a) is estimable since $\lambda = (1, 1, 0, ...0)^T \in C(\mathbf{X}^T)$, in fact $\lambda = \mathbf{x}_1^T$.

(b) is not estimable since $\lambda = (1, 0, ..., 0)^T \notin C(\mathbf{X}^T)$. To see this, try taking any linear combination of the rows of $\mathbf{X}$ and you will always have a 1 remaining after the first index of the resulting vector.

(c) is not estimable since $\lambda = (0, 0, 1, , ..., 1)^T \notin C(\mathbf{X}^T)$. To see this, try taking any linear combination of the rows of $\mathbf{X}$, but we can only obtain a vector like $(a, 0, 1, ..., 1)^T$, not $\lambda$.

(d) is estimable since it is a contrast.